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A COHOMOLOGY DECOMPOSITION THEOREM

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§1. INTRODUCTION

In [9] Jackowski and McClure gave a homotopy decomposition theorem for the classifying space of a compact Lie group G ; their theorem states that for any prime p the space BG can be constructed at p as the homotopy direct limit of a specific diagram involving the classifying spaces of centralizers of elementary abelian p -subgroups of G . In this paper we will prove a parallel algebraic decomposition theorem for certain kinds of unstable algebras over the mod p Steenrod algebra. This algebraic result gives a new proof of the theorem of Jackowski and McClure and has the potential to lead to homotopy decomposition theorems for many spaces which are not of the form BG (see §6).

Before stating our results we will recall some material from [9]. Choose a prime p . Let G be a compact Lie group, and let A_G be the category whose objects are the non-trivial elementary abelian p -subgroups of G ; a morphism $A \rightarrow A'$ in A_G is a homomorphism $f: A \rightarrow A'$ of abelian groups with the property that there exists an element $g \in G$ such that $f(x) = gxg^{-1}$ for all $x \in A$. There is a functor from A_G^{op} to the category of topological spaces which sends A to the Borel construction $EG \times_G (G/C(A))$, where $C(A)$ denotes the centralizer of A in G . (Notice that this Borel construction has the homotopy type of the classifying space $BC(A)$.) Jackowski and McClure prove that the natural map from the homotopy direct limit of this functor to $EG \times_G * = BG$ is an isomorphism on mod p cohomology. They derive this from a spectral sequence argument [2, XII, 5.8] and the following calculation with the inverse limit functor \varprojlim and its right derived functors \varprojlim^i . Let H^* denote mod p cohomology and α_G the functor on A_G which sends A to $H^*(EG \times_G (G/C(A)))$.

THEOREM 1.1. [9, Prop. 3–4]. *The natural map $H^* BG \rightarrow \varprojlim \alpha_G$ is an isomorphism and the groups $\varprojlim^i \alpha_G$ vanish for $i > 0$.*

The proof of Theorem 1.1 in [9] uses the Feshbach double coset formula and so depends heavily on the presence of a genuine compact Lie group. What we do is much more algebraic. Let \mathcal{X} be the category of unstable algebras over the mod p Steenrod algebra \mathcal{A}_p . Given an object R of \mathcal{X} we will build an index category A_R together with a functor $\alpha_R: A_R \rightarrow \mathcal{X}$ and natural map $R \rightarrow \varprojlim \alpha_R$; if $R = H^* BG$ then A_R is equivalent to A_G in such a way that α_R corresponds to α_G . This construction depends heavily on work of Lannes [11]. Using [11] again, we will define what it means for an object R of \mathcal{X} to “have a non-trivial center”; if $R = H^* BG$ this condition holds if G has a non-trivial central element of order p . Our main result is the following one.

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THEOREM 1.2. Suppose that $i: R \rightarrow S$ is a map of \mathcal{K} such that:

- (1) Both R and S are finitely generated as algebras, and the map i makes S into a finitely generated module over R .
- (2) The map i has an additive left inverse $S \rightarrow R$ which is both a map of R modules and a map of unstable modules over the Steenrod algebra.
- (3) The algebra S has a non-trivial center.

Then the natural map $R \rightarrow \varprojlim \alpha_R$ is an isomorphism and the groups $\varprojlim^i \alpha_R$ vanish for $i > 0$.

Remark. For a functor such as α_R which takes values in the category \mathcal{K} , we write $\varprojlim^i \alpha_R$ ($i > 0$) for the ordinary higher limits [2, p. 305] of the composite of α_R with the forgetful functor from \mathcal{K} to the category of graded \mathbb{F}_p vector spaces.

The connection between Theorem 1.2 and Theorem 1.1 is provided by the following proposition, which lists some standard properties of compact Lie groups.

PROPOSITION 1.3. Let G be a compact Lie group, T a maximal torus in G , $N(T)$ the normalizer of T , $N_p(T)$ the inverse image in $N(T)$ of a p -Sylow subgroup of $N(T)/T$, and i^* the natural restriction map $H^*BG \rightarrow H^*BN_p(T)$. Then the following assertions hold.

- (1) Both H^*BG and $H^*BN_p(T)$ are finitely generated algebras [17] [15, 2.2]. The map i^* makes $H^*BN_p(T)$ into a finitely generated module over H^*BG [15, 2.4].
- (2) The map i^* has an additive left inverse $H^*BN_p(T) \rightarrow H^*BG$ which is both a H^*BG module map and a map of unstable modules over the mod_p Steenrod algebra [1].
- (3) The group $N_p(T)$ has a non-trivial central element of order p .

Remark 1.4. Note that 1.3(3) follows from the fact that the conjugation action of $N_p(T)/T$ on the elements of order p in T must pointwise fix a non-trivial subgroup [8, p. 47].

There is another basic example of the situation described in Theorem 1.2, an example which is purely algebraic. Suppose that τ is either a torus or an elementary abelian p -group. Let W be a finite group of \mathcal{K} -automorphisms of $H^*B\tau$ and W_p a p -Sylow subgroup of W . Then the rings of invariants $R = H^*(B\tau)^W$ and $S = H^*(B\tau)^{W_p}$ are finitely generated as algebras and the natural inclusion $R \rightarrow S$ has a left inverse given by averaging over the coset space W/W_p . It is not hard to see that S has a non-trivial center (cf. 1.4, 3.4 and the definition in §4 of “having a non-trivial center”). This leads to

PROPOSITION 1.5. Let τ be either a torus or an elementary abelian p -group. Suppose that W is a finite group of \mathcal{K} -automorphisms of $H^*B\tau$ and that R is the ring of invariants $H^*(B\tau)^W$. Then the natural map $R \rightarrow \varprojlim \alpha_R$ is an isomorphism and the groups $\varprojlim^i \alpha_R$ vanish for $i > 0$.

ORGANIZATION OF THE PAPER: Section 2 constructs the decomposition functor α_R and compares it with α_G if $R = H^*BG$. Section 3 describes some properties of Lannes’s functor T . In §4 there is a definition of what it means for an object R of \mathcal{K} to “have a non-trivial center” and a proof of (a slight generalization of) the special case of 1.2 in which $R = S$. Section 5 completes the proof of 1.2 itself, and §6 concludes the paper with a topological application of 1.5.

NOTATION AND TERMINOLOGY: The prime p will be fixed for the rest of the paper. If V is an elementary abelian p -group (i.e. a finite dimensional \mathbb{F}_p vector space) then H^V will stand for

the mod p cohomology algebra H^*BV . At several points in the paper we will speak for convenience of inverse limits and related constructions over certain large categories. In each case the large category in question is equivalent to a small subcategory of itself (i.e. the category has a small skeleton [13, p. 91]) and the constructions can be performed in the usual way on these small subcategories.

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§2. THE DECOMPOSITION DIAGRAM

Given an object R of \mathcal{X} , define V_R to be the category whose objects consist of pairs (V, f) , where V is an elementary abelian p -group and $f: R \rightarrow H^V$ is a \mathcal{X} -map. A map $(V, f) \rightarrow (W, g)$ in V_R is an abelian group map $h: V \rightarrow W$ such that the composite of the induced map $h^*: H^W \rightarrow H^V$ with g is f . We will usually write an object (V, f) of V_R as $f: V \rightsquigarrow R$ or just $V \rightsquigarrow R$; in this notation V is identified with the object H^V of \mathcal{X} and " \rightsquigarrow " denotes a morphism of \mathcal{X}^{op} . A morphism from $f: V \rightsquigarrow R$ to $g: W \rightsquigarrow R$ is then a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & R \\ h \downarrow & & \parallel \\ W & \xrightarrow{g} & R \end{array}$$

in which "commutativity", of course, means $h^*g = f$; sometimes we will even refer to f as the "composite" of g and h .

The category A_R is a subcategory of V_R . An object $f: V \rightsquigarrow R$ is in A_R if $V \neq \{0\}$ and f makes H^V into a finitely generated module over R ; a map h as above is in A_R if the abelian group map $h: V \rightarrow W$ is a *monomorphism*.

Recall from [10, 3.4] that for any elementary abelian p -group V there is a functor $T^V: \mathcal{X} \rightarrow \mathcal{X}$ left adjoint to the functor given by tensor product over F_p with H^V ; moreover [4], for any fixed $f: V \rightsquigarrow R$ there is a distinguished quotient $T_f^V(R)$ of $T^V R$. We will alter the standard notation slightly and write $T(V, R)$ instead of $T^V R$ and $T(V, R)_f$ instead of $T_f^V R$. A homomorphism $h: V \rightarrow W$ of elementary abelian p -groups induces a natural transformation $T(h, -): T(V, -) \rightarrow T(W, -)$; given $f: V \rightsquigarrow R$ and $g: W \rightsquigarrow R$ with $h^*g = f$, the map $T(h, R): T(V, R) \rightarrow T(W, R)$ passes to a quotient map $T(V, R)_f \rightarrow T(W, R)_g$ (see 3.1). Define the functor $\alpha_R: A_R \rightarrow \mathcal{X}$ by setting $\alpha_R(f: V \rightsquigarrow R)$ equal to $T(V, R)_f$. For any V the inclusion $0 \subset V$ induces a composite map $\varepsilon: R = T(0, R) \rightarrow T(V, R) \rightarrow T(V, R)_f$ (cf. [4, §2]); these combine to give a map $R \rightarrow \varprojlim \alpha_R$.

Remark 2. If $R = H^*X$ for some space X , then under mild assumptions [10] $T(V, R)$ is naturally isomorphic to the cohomology of the mapping space $\text{Map}(BV, X)$ and $T(V, R)_f$ to the cohomology of the subspace $\text{Map}(BV, X)_f$ consisting of maps which induce f on cohomology. If $R = H^*BG$ for a compact Lie group G , then $f: V \rightsquigarrow R$ corresponds to a homomorphism $\bar{f}: V \rightarrow G$ (unique up to conjugacy) and $T(V, R)_f$ is naturally isomorphic to $H^*BC(\text{im}(\bar{f}))$ (see [11]). The connection implied here between mapping spaces and centralizers is made explicit in [6]. Note that [15, 2.4] a homomorphism $V \rightarrow G$ is a monomorphism if and only if the induced map $V \rightsquigarrow H^*BG$ makes H^V into a finitely generated module over H^*BG .

PROPOSITION 2.2. *If $R = H^*BG$ for a compact Lie group G , then there is an equivalence of categories $e: A_G \rightarrow A_R$ such that the composite functor $\alpha_R \cdot e$ is naturally equivalent to α_G .*

Proof. Define e to be the functor which sends a non-trivial elementary abelian p -subgroup A of G to the pair (A, j_A) , where $i_A: BA \rightarrow BG$ is the map induced by inclusion and $j_A = i_A^*$; as above the map j_A makes H^A into a finitely generated module over H^*BG . If $h: A \rightarrow B$ is a map between two elementary abelian subgroups of G which is realized by conjugation with an element of G , then $j_A = h^*j_B$ because inner automorphisms of G induce the identity map on cohomology. Since homomorphisms between elementary abelian p -groups are detected in cohomology, the functor e is faithful. The fact that e is full and has image intersecting every isomorphism class in \mathbf{A}_R follows from the result of [11] referred to in 2.1. By [13, p. 91], then, the functor e is an equivalence of categories.

A map $h: A \rightarrow B$ in \mathbf{A}_G can be represented by an element $x_h \in G$ which conjugates A in a specific way to a subgroup of B . It is clear that x_h is unique up to right multiplication by elements of $C(A)$. Conjugation with x_h^{-1} then carries $C(B)$ to $C(A)$ in a way which is well-defined up to inner automorphisms of $C(A)$. Since any inner automorphism of a group induces a self-map of the classifying space which is homotopic to the identity [15, p. 551], we can construct a functor F from \mathbf{A}_G^{op} to the homotopy category of spaces by setting $F(A) = BC(A)$; it is easy to check that up to homotopy this is the same as the functor [9] which assigns to A the space $EG \times_G (G/C(A)) = EG/C(A)$. Consequently, the functor α_G is naturally equivalent to H^*F . The multiplication map $\mu_A: BA \times BC(A) \rightarrow BG$ induces a \mathcal{X} -map $H^*BG \rightarrow H^A \otimes H^*BC(A)$ which has as adjoint an isomorphism [11] $\alpha_R(e(A)) = T(A, H^*BG)_{j_A} \rightarrow H^*BC(A) = H^*F(A)$. The fact that these isomorphisms combine to give a natural equivalence $\alpha_R \cdot e \rightarrow H^*F$ follows from the fact that for any $h: A_1 \rightarrow A_2$ in \mathbf{A}_G the diagram

$$\begin{array}{ccc} BA_1 \times BC(A_2) & \xrightarrow{1 \times x_h^{-1}(\cdot)x_h} & BA_1 \times BC(A_1) \\ \downarrow h \times 1 & & \downarrow \mu_{A_1} \\ BA_2 \times BC(A_2) & \xrightarrow{\mu_{A_2}} & BG \end{array}$$

commutes up to homotopy.

It will be convenient below to work with a functor β_R which is closely related to α_R . Let \mathbf{A} denote the category whose objects are non-trivial elementary abelian p -groups and whose morphisms are monic group homomorphisms. For any object R of \mathcal{X} , let β_R denote the functor $\mathbf{A} \rightarrow \mathcal{X}$ which assigns to V the product $\prod_f T(V, R)_f$, where f runs through all $V \hookrightarrow R$ which make H^V into a finitely generated R module. If $j: V \rightarrow W$ is a map of \mathbf{A} (i.e. a monomorphism), there is for each such $f: V \hookrightarrow R$ a natural map $T(V, R)_f \rightarrow \prod_g T(W, R)_g$, where this latter product is indexed by $g: W \hookrightarrow R$ such that g makes H^W into a finitely generated R module and $j^*g = f$. Combining these maps for all such f gives a map $\beta_R(V) \rightarrow \beta_R(W)$. The natural maps $\varepsilon: R \rightarrow T(V, R)_f$ combine to give a natural map $R \rightarrow \varprojlim \beta_R$.

PROPOSITION 2.3. *There are natural isomorphisms $\varprojlim^i \alpha_R \rightarrow \varprojlim^i \beta_R$ for all $i \geq 0$. Under these isomorphisms the natural map $R \rightarrow \varprojlim \alpha_R$ corresponds to the natural map $R \rightarrow \varprojlim \beta_R$.*

Proof. There is a forgetful functor $\Phi: \mathbf{A}_R \rightarrow \mathbf{A}$ which assigns to an object (V, f) of \mathbf{A}_R the underlying vector space V . Associated to this is a composition of functors spectral sequence

$$E_{p,q}^2 = \varprojlim^p \Phi^q(\alpha_R) \Rightarrow \varprojlim^{p+q} \alpha_R$$

in which the functors $\Phi^q(x_R)$ are defined as follows. For $V \in \mathbf{A}$, let $V \downarrow \Phi$ be the under category [13, p. 46] and x_R^V the functor obtained by composing x_R with the forgetful functor $V \downarrow \Phi \rightarrow \mathbf{A}_R$. Then $\Phi^q(x_R)(V)$ is $\lim_{\leftarrow}^q x_R^V$. (For a dual (direct limit) description of this spectral sequence see [7, pp. 155–157].) To complete the proof it is enough to show that $\Phi^0(x_R)$ is isomorphic to β_R and that $\Phi^q(x_R)$ vanishes for $q > 0$. For $V \in \mathbf{A}$ let $\Phi^{-1}V$ denote the subcategory of \mathbf{A}_R consisting of objects A such that $\Phi(A) = V$; a morphism in $\Phi^{-1}V$ is a morphism in \mathbf{A}_R which projects under Φ to the identity map of V . A short calculation shows that the evident inclusion map $j: \Phi^{-1}V \rightarrow V \downarrow \Phi$ is left cofinal [2, XI, §9] so that the induced maps $\lim_{\leftarrow}^q x_R^V \rightarrow \lim_{\leftarrow}^q x_R^V \circ j$ are isomorphisms (cf. [2, XI, 7.2 and 9.2]). The desired result now follows from the fact that $\Phi^{-1}V$ is equivalent to a discrete category, that is, a category with no non-identity morphisms.

Remark. The functor β_R is the right Kan extension [13, p. 232] of x_R along the forgetful functor Φ which appears in the above proof. Proposition 2.3 can also be deduced from [9, 3.1].

§3. PROPERTIES OF T

In this section we will describe some properties of the functor T which are used in §4 and §5. Most of these properties are algebraic analogs of simple properties of function spaces (cf. 2.1). Throughout the section R will stand for an object of \mathcal{X} and V for an elementary abelian p -group.

Recall [4, §2] that $T(V, R)_f$ is the tensor product $\mathbf{F}_p \otimes_{T(V, R)^0} T(V, R)$, where the composite map $T(V, R) \rightarrow T(V, R)^0 \rightarrow \mathbf{F}_p$ is adjoint to $f: R \rightarrow H^V$. Let $I(R) \subset R$ denote the ideal of positive dimensional elements.

LEMMA 3.1. *Let $f: V \rightsquigarrow R$ be an object of \mathbf{V}_R , and S an object of \mathcal{X} with unit inclusion $\eta: \mathbf{F}_p \rightarrow S^0$ and projection $\pi: S \rightarrow S/I(S) = S^0$. Then the set $\text{Hom}_{\mathcal{X}}(T(V, R)_f, S)$ is naturally isomorphic to the set of maps $g: R \rightarrow H^V \otimes S$ such that $(1 \otimes \pi) \cdot g$ is equal to the composite $(1 \otimes \eta) \cdot f: R \rightarrow H^V = H^V \otimes \mathbf{F}_p \rightarrow H^V \otimes (S^0)$.*

Proof. This follows immediately from the tensor product formula for $T(V, R)_f$ and the defining adjointness property of $T(V, -)$.

LEMMA 3.2. *Given $f: V \rightsquigarrow R$, the natural map $\varepsilon: R \rightarrow T(V, R)_f$ [4, §2] induces via 3.1 the map which assigns to $R \rightarrow H^V \otimes S$ the composite $R \rightarrow H^V \otimes S \rightarrow S$ obtained using the projection $H^V \rightarrow \mathbf{F}_p$.*

Proof. This follows immediately from [4, §2] and could in fact be used to define ε .

PROPOSITION 3.3. *Given $f: V \oplus W \rightsquigarrow R$, let $a: V \rightsquigarrow R$ and $b: W \rightsquigarrow R$ be the composites of f with the appropriate direct summand inclusions, and let $g: W \rightsquigarrow T(V, R)_a$ be the map which corresponds via 3.1 to $f: R \rightarrow H^V \otimes H^W \cong H^{V \oplus W}$. Then there is a natural isomorphism $T(W, T(V, R)_a)_g \rightarrow T(V \oplus W, R)_f$. Under this isomorphism the map $T(W, R)_b \rightarrow T(V \oplus W, R)_f$ induced by $W \rightarrow V \oplus W$ corresponds to the map $T(W, R)_b \rightarrow T(W, T(V, R)_a)_g$ obtained by applying $T(W, -)$ to the natural map $\varepsilon: R \rightarrow T(V, R)_a$.*

Proof. This is a routine application of Yoneda's lemma [13, p. 62, Ex. 2] that involves using 3.1 and 3.2 to identify the functors corepresented by the objects $T(W, R)_b$, $T(V \oplus W, R)_f$, and $T(W, T(V, R)_a)_g$.

Recall [12] that a module M over the mod p Steenrod algebra \mathcal{A}_p is said to be *locally finite* if every element of M is contained in a finite \mathcal{A}_p submodule. Let $Q(R)$ denote the quotient $I(R)/I(R)^2$.

PROPOSITION 3.4. *Let $f: V \rightsquigarrow R$ be an object of \mathbf{V}_R , and assume that $Q(R)$ is locally finite as a module over \mathcal{A}_p . Then the natural map $R \rightarrow T(V, R)_f$ is an isomorphism iff*

- (1) R is connected (i.e. $R^0 \cong \mathbb{F}_p$) and
- (2) there exists a \mathcal{X} -map $R \rightarrow H^V \otimes R$ which, composed with the evident projections $H^V \otimes R \rightarrow R$ and $H^V \otimes R \rightarrow H^V$, gives, respectively, the identity map of R and the map f .

Proof. This follows from [5, 4.7, Proof of 4.1].

§4. CENTERS

In this section we will define what it means for an object R of \mathcal{X} to have a non-trivial center and then prove a special case (Proposition 4.10) of Theorem 1.2. Throughout the section R will stand for an object of \mathcal{X} and V for an elementary abelian p -group. We will develop a formal analogy (cf. 2.1) between properties of the objects $V \rightsquigarrow R$ for $R \in \mathcal{X}$ and properties of homomorphisms up to conjugacy $V \rightarrow G$ for a compact Lie group G . It will become clear below that this analogy works well only if $Q(R)$ is locally finite as a module over \mathcal{A}_p . Once the analogy is set up, the proof of 4.10 can be carried out by mimicking a group theoretic argument.

Definition 4.1. An object $f: V \rightsquigarrow R$ of \mathbf{V}_R is said to be

- (1) *monic*, if f makes H^V into a finitely generated module over R , and
- (2) *central*, if the natural map $R \rightarrow T(V, R)_f$ is an isomorphism.

The algebra R is said to *have a non-trivial center* if there exists a monic central map $V \rightsquigarrow R$ with $V \neq \{0\}$.

Recall from 2.1 that if $R = H^*BG$ for a compact Lie group G then an object $f: V \rightsquigarrow R$ of \mathbf{V}_R corresponds to a conjugacy class f of group homomorphisms $V \rightarrow G$. The object f is *monic* in the above sense iff (any representative of) f is a group monomorphism, and *central* if the image of \bar{f} lies in the center of G . In particular, the following is true.

PROPOSITION 4.2. *If G is a compact Lie group with a non-trivial central element of order p , then H^*BG has a non-trivial center.*

In 4.3–4.9 below we will prove in a more general setting statements which are obvious from the above remarks in the special case $R = H^*BG$.

Let $f: V \rightsquigarrow R$ be an object of \mathbf{V}_R . Choosing an element v of V amounts to giving a homomorphism $\chi_v: \mathbb{Z}/p \rightarrow V$ with $\chi_v(1) = v$; we will let $f_v: \mathbb{Z}/p \rightsquigarrow R$ stand for the composite $\mathbb{Z}/p \xrightarrow{\chi_v} V \rightsquigarrow R$.

Definition 4.3. An object $f: V \rightsquigarrow R$ of \mathbf{V}_R is said to be *null* if the \mathcal{X} -map $f: R \rightarrow H^V$ is trivial above dimension 0. The *kernel* of f , denoted $\ker(f)$, is the set consisting of all elements $v \in V$ with the property that $f_v: \mathbb{Z}/p \rightsquigarrow R$ is null.

PROPOSITION 4.4. An object $f: V \rightsquigarrow R$ of \mathbf{V}_R is monic iff $\ker(f) = \{0\}$.

Proof. If $\ker(f)$ is not trivial, there is a surjective map $g: H^V \rightarrow H^{\mathbb{Z}/p}$ such that the composite gf is trivial above dimension 0; since $H^{\mathbb{Z}/p}$ is not of finite rank as an \mathbb{F}_p vector space, this implies that H^V is not finitely generated as an R module, i.e. that f is not monic. Conversely, if H^V is not finitely generated as an R module then the quotient ring $\mathbb{F}_p \otimes_R H^V$ is not finite-dimensional over \mathbb{F}_p . This quotient ring is generated as an algebra by exterior generators $\{y_i\}$ of dimension 1 together with their Bockstein images $\{\beta y_i\}$ in dimension 2. If the quotient ring is not finite dimensional, there must be some $y \in \{y_i\}$ such that βy is not nilpotent. The subring S of $\mathbb{F}_p \otimes_R H^V$ generated by y and βy is clearly closed under the action of \mathcal{A}_p and isomorphic as an \mathcal{A}_p algebra to $H^{\mathbb{Z}/p}$. By the injectivity of $H^{\mathbb{Z}/p}$ as an \mathcal{A}_p algebra [10, 3.6], any chosen isomorphism $S \rightarrow H^{\mathbb{Z}/p}$ extends to a \mathcal{X} -map $\mathbb{F}_p \otimes_R H^V \rightarrow H^{\mathbb{Z}/p}$; the composite of such an extension with the quotient map $H^V \rightarrow \mathbb{F}_p \otimes_R H^V$ is a \mathcal{X} map $H^V \rightarrow H^{\mathbb{Z}/p}$ representing [10, 4.2] a non-trivial element of V in $\ker(f)$.

LEMMA 4.5. Suppose that $R \in \mathcal{X}$ is connected and that $Q(R)$ is locally finite as a module over \mathcal{A}_p . Then any null object $f: V \rightsquigarrow R$ is central. If $g: C \rightsquigarrow R$ is central and C' is a subgroup of C then the composite object $C' \rightarrow C \rightsquigarrow R$ is also central.

Proof. This is a consequence of the characterization of central objects in Proposition 3.4.

LEMMA 4.6. Let $f: V \rightsquigarrow R$ and $g: C \rightsquigarrow R$ be objects of \mathbf{V}_R , and assume that g is central. Then there is a unique object $f \oplus g: V \oplus C \rightsquigarrow R$ which restricts to f (resp. g) along the summand inclusion $V \rightarrow V \oplus C$ (resp. $C \rightarrow V \oplus C$).

Proof. The desired object $V \oplus C \rightsquigarrow R$ corresponds to a map $h: R \rightarrow H^{V \oplus C} \cong H^V \otimes H^C$ which agrees with f when composed with $H^V \otimes H^C \rightarrow H^V$ and with g when composed with $H^V \otimes H^C \rightarrow H^C$. According to 3.1 and 3.2, such an h amounts to a map $\bar{h}: T(C, R)_g \rightarrow H^V$ which agrees with f when composed with $\varepsilon: R \rightarrow T(C, R)_g$. The lemma follows from the fact that, since g is central, the map $\varepsilon: R \rightarrow T(C, R)_g$ is an isomorphism.

LEMMA 4.7. Suppose that $R \in \mathcal{X}$ is connected and that $Q(R)$ is locally finite as a module over \mathcal{A}_p . Let $f: V \rightsquigarrow R$ be an object of \mathbf{V}_R , $v \in V$ an element of $\ker(f)$, and $\langle v \rangle$ the subgroup of V generated by v . Then f extends to a unique map $g: V/\langle v \rangle \rightsquigarrow R$.

Proof. Let $W \subset V$ be a complement to $\langle v \rangle$ and $h: W \rightsquigarrow R$ the composite of f with the inclusion $W \rightarrow V$. There is an isomorphism $V \cong W \oplus \langle v \rangle$ and by 4.5 the composite $\langle v \rangle \rightarrow V \rightsquigarrow R$ is central; by 4.6, then, f is the unique object $V \rightsquigarrow R$ which is null on $\langle v \rangle$ and agrees with h on W . Clearly, then, f is the composite of h with the projection map $V \rightarrow V/\langle v \rangle \cong W$.

Remark. The following example shows that in Lemma 4.7 some restriction on R is necessary. Let V be an elementary abelian p -group of rank greater than 1, $J \subset H^V$ the ideal

generated by the product $\prod_y \beta_y$ as y runs through the non zero one-dimensional elements of H^V , R the ring $\mathbb{F}_p \oplus J$, and $f: R \rightarrow H^V$ the evident inclusion. Every element of V is in the kernel of $f: V \rightsquigarrow R$ but f does not extend over V/W for any non-trivial subgroup W of V .

PROPOSITION 4.8. *Suppose that $R \in \mathcal{K}$ is connected and that $Q(R)$ is locally finite as module over \mathcal{A}_p . Let $f: V \rightsquigarrow R$ be an object of \mathbf{V}_R . Then $\ker(f)$ is a subgroup of V , and f extends uniquely to a monic map $g: V/\ker(f) \rightsquigarrow R$.*

Proof. Let W be a maximal subgroup of V such that f extends to an object $g: V/W \rightsquigarrow R$. It is clear that $W \subset \ker(f)$. However, g must be monic by 4.7, so it follows easily that $\ker(f) \subset W$.

PROPOSITION 4.9. *Suppose that $R \in \mathcal{K}$ is connected and that $Q(R)$ is locally finite as a module over \mathcal{A}_p . Let $f: V \rightsquigarrow R$, $g: C \rightsquigarrow R$, and $h: W \rightsquigarrow R$ be objects of \mathbf{V}_R with g central and h monic, and let $f \oplus g: V \oplus C \rightsquigarrow R$ be the unique object (4.6) which extends f and g . Then, given a map $u: f \rightarrow h$ in \mathbf{V}_R , there is at most one way of extending u to a map $\hat{u}: f \oplus g \rightarrow h$.*

Proof. The map \hat{u} amounts to an abelian group map $V \oplus C \rightarrow W$ which is prescribed on V ; to show that \hat{u} is unique (if it exists) it is enough to treat the special case $V = \{0\}$ and (see 4.5) $C \cong \mathbb{Z}/p$. Suppose that \hat{u}_1 and \hat{u}_2 are two maps $\mathbb{Z}/p \rightarrow W$ which give the same object $g: \mathbb{Z}/p \rightsquigarrow R$ when composed with $h: W \rightsquigarrow R$, and let $\hat{u}_1 + \hat{u}_2: \mathbb{Z}/p \oplus \mathbb{Z}/p \rightarrow W$ be their sum. The composite of $\hat{u}_1 + \hat{u}_2$ with h is an object $\mathbb{Z}/p \oplus \mathbb{Z}/p \rightsquigarrow R$ which by 4.6 agrees with the composite $\mathbb{Z}/p \oplus \mathbb{Z}/p \xrightarrow{\hat{u}} \mathbb{Z}/p \rightsquigarrow R$ and therefore has each element of the form $(x, -x)$ in $\mathbb{Z}/p \oplus \mathbb{Z}/p$ in its kernel. Since h is monic, it follows easily that $\ker(\hat{u}_1 + \hat{u}_2)$ also contains all elements of the form $(x, -x)$. This shows that $\hat{u}_1(x) = \hat{u}_2(x)$ for all $x \in \mathbb{Z}/p$.

PROPOSITION 4.10. *Suppose that R is an object of \mathcal{K} with the property that $Q(R)$ is locally finite as a module over \mathcal{A}_p . Assume that R has a non-trivial center. Then the natural map $R \rightarrow \varprojlim \alpha_R$ is an isomorphism and the groups \varprojlim^i vanish for $i > 0$.*

Remark. The proof of 4.10 shows that if R has a non-trivial center then the nerve of the category \mathbf{A}_R is contractible. In fact, the main argument in this proof is very similar to the argument in Quillen's proof that the poset of elementary abelian subgroups of a finite group is contractible if the group has a non-trivial normal p -subgroup [16, 2.4].

Proof of 4.10. Observe by 3.4 that R is connected. Choose a monic central object $g: C \rightsquigarrow R$ of \mathbf{V}_R such that $C \neq \{0\}$, so that g also represents an object of \mathbf{A}_R . Let $g \downarrow \mathbf{A}_R$ be the under category [13, p. 46] and $j: g \downarrow \mathbf{A}_R \rightarrow \mathbf{A}_R$ the forgetful functor. The category $g \downarrow \mathbf{A}_R$ has the identity map of g as an initial object and by assumption the natural map $R \rightarrow \alpha_R(g) = \alpha_R(j(g \xrightarrow{1} g))$ is an isomorphism; it follows [2, XI, 7.2 and 9.2] that the natural map $R \rightarrow \varprojlim (\alpha_R \cdot j)$ is an isomorphism and that $\varprojlim^i (\alpha_R \cdot j)$ vanishes for $i > 0$. Given an arbitrary object $f: V \rightsquigarrow R$ of \mathbf{A}_R there is (4.6) a unique map $f \oplus g: V \oplus C \rightsquigarrow R$ of \mathbf{V}_R which restricts to f (resp. g) along the summand inclusion $V \rightarrow V \oplus C$ (resp. $C \rightarrow V \oplus C$); let $\sigma(f): (V \oplus C)/\ker(f \oplus g) \rightsquigarrow R$ be the corresponding (4.8) object of \mathbf{A}_R . It is clear that the construction $f \mapsto \sigma(f)$ produces a functor $\sigma: \mathbf{A}_R \rightarrow g \downarrow \mathbf{A}_R$ and that the natural map $V \rightarrow (V \oplus C)/\ker(f \oplus g)$ induces a natural transformation τ from the identity functor of \mathbf{A}_R to the composite $j \cdot \sigma$. Now the group $(V \oplus C)/\ker(f \oplus g)$ can be written as a direct sum $V \oplus C'$, where $C' \subset C$ is complementary to the kernel of the composite map

$C \rightarrow V \oplus C \rightarrow (V \oplus C)/(V + \ker(f \oplus g))$; by (4.5) the composite $C' \rightarrow C \rightsquigarrow R$ is central, and so by (3.3) the map $T(V, R)_f \rightarrow T((V \oplus C)/\ker(f \oplus g), R)_{\sigma(f)}$ induced by $V \rightsquigarrow (V \oplus C)/\ker(f \oplus g)$ is an isomorphism. It follows that the natural transformation from α_R to $\alpha_R \cdot j \cdot \sigma$ induced by τ is a natural equivalence.

Now let $x = (C \rightarrow W \rightsquigarrow R)$ be an object of $g \downarrow \mathbf{A}_R$, let $f: V \rightsquigarrow R$ be an object of \mathbf{A}_R , and let $h: W \rightsquigarrow R$ in \mathbf{A}_R be the image $j(x)$ of x under the forgetful functor. Suppose that $\sigma(f) \rightarrow x$ is a map in $g \downarrow \mathbf{A}_R$. According to (4.9) and the above remarks there is a unique map $\omega: f \rightarrow h$ in \mathbf{A}_R such that the given map $\sigma(f) \rightarrow x$ is the composite of $\sigma(\omega)$ with the evident natural isomorphism $\sigma(j(x)) = \sigma(h) \cong x$. This shows that the over category $\sigma \downarrow x$ has a terminal object and therefore a contractible nerve [2, XI, §2]; consequently, the functor σ is left cofinal [2, XI, §9]. The proposition now follows directly from the fact [2, XI, 7.2. and 9.2] that the natural maps $\varprojlim^i (\alpha_R \cdot j) \rightarrow \varprojlim^i (\alpha_R \cdot j \cdot \sigma) \cong \varprojlim^i \alpha_R$ are isomorphisms for all i .

§5. COMPLETION OF THE PROOF

In this section we will complete the proof of Theorem 1.2 by showing that the functor β_R of 2.3 is a retract of the acyclic (4.10) functor β_S .

Note that if R is an object of \mathcal{X} which is finitely generated as an algebra and V is an elementary abelian p -group, there are only a finite number of \mathcal{X} -maps $R \rightarrow H^V$ (i.e. objects $V \rightsquigarrow R$). In this case $T(V, R)$ splits as a direct product $\prod_f T(V, R)_f$ indexed by $f: V \rightsquigarrow R$ (see [10, 3.5]). Recall from §2 that there is a functor $\beta_R: \mathbf{A} \rightarrow \mathcal{X}$ which assigns to each elementary abelian p -group V the product $\prod_f T(V, R_f)$ where f runs through *monic* (§4) objects $V \rightsquigarrow R$, (that is, maps $f: R \rightarrow H^V$ which make H^V into a finitely generated module over R).

LEMMA 5.1. *Suppose that $i: R \rightarrow S$ is a map of \mathcal{X} such that R and S are finitely generated as algebras and i makes S into a finitely generated module over R . Then for any elementary abelian p -group V the quotient $\beta_S(V)$ of $T(V, S)$ is naturally isomorphic to the tensor product $\beta_R(V) \otimes_{T(V, R)} T(V, S)$.*

Proof. For fixed $f: R \rightarrow H^V$, the tensor product $T(V, R)_f \otimes_{T(V, R)} T(V, S)$ is a direct product $\prod_g T(V, S)_g$ where g runs through all maps $S \rightarrow H^V$ which extend f . The lemma follows from the fact that such a g makes H^V into a finitely generated module over S iff f makes H^V into a finitely generated module over R .

Remark 5.2. Let \mathcal{U} be the category of unstable modules over \mathcal{A}_p and $\Phi: \mathcal{X} \rightarrow \mathcal{U}$ the forgetful functor [10, §1]. There is a functor $T^\#(V, -): \mathcal{U} \rightarrow \mathcal{U}$ defined like $T(V, -)$ such that for $R \in \mathcal{X}$ the module $T^\#(V, \Phi(R))$ is naturally isomorphic to $\Phi(T(V, R))$ (see [10, 3.4]). If M in \mathcal{U} is a module over R in such a way that the Cartan formula holds for module multiplication (write $M \in \mathcal{U}(R)$ [4, §1]) then $T^\#(V, M)$ is a module over $T(V, R)$ [10, 3.2.1]. Finally, if $i: R \rightarrow S$ is a map of \mathcal{X} which is used to make $\Phi(S)$ into a module over R , then the $T(V, R)$ module structure on $T^\#(V, \Phi(S))$ described above agrees via the natural isomorphism $T^\#(V, \Phi(S)) \cong \Phi(T(V, S))$ with the module structure on $T(V, S)$ derived from the induced ring homomorphism $T(V, R) \rightarrow T(V, S)$.

Proof of 1.2. Let $\gamma_R: \mathbf{A} \rightarrow \mathcal{X}$ assign to the elementary abelian p -group V the ring $T(V, R)$ and let $\gamma_S: \mathbf{A} \rightarrow \mathcal{X}$ assign the ring $T(V, S)$. The functors β_R, β_S and γ_S are modules over γ_R (i.e., for any V in \mathbf{A} the object $\gamma_R(V)$ is a ring and the objects $\beta_R(V), \beta_S(V)$ and $\gamma_S(V)$ are modules over $\gamma_R(V)$ in a way that varies in an evident functorial way with V). By 5.1 there is a natural equivalence $\beta_S \cong \beta_R \otimes_{\gamma_R} \gamma_S$. By 5.2 the $\mathcal{U}(R)$ -map $S \rightarrow R$, left inverse to $R \rightarrow S$,

induces a map

$$\beta_S \cong \beta_R \otimes_{\gamma_R \gamma_S} \gamma_S \rightarrow \beta_R \otimes_{\gamma_R \gamma_R} \gamma_R \cong \beta_R$$

left inverse to the natural map $\beta_R \rightarrow \beta_S$. By 4.10 and 2.3 the natural map $S \rightarrow \varprojlim \beta_S$ is an isomorphism and $\varprojlim^i \beta_S = 0$ for $i > 0$. It follows easily that $R \rightarrow \varprojlim \beta_R$ is an isomorphism (any retract of an isomorphism is an isomorphism) and that $\varprojlim^i \beta_R = 0$ for $i > 0$. The conclusion of the theorem results from another application of 2.3.

§6. A HOMOTOPY DECOMPOSITION THEOREM

In this section we will show that our main algebraic results give rise to homotopy-theoretic decompositions in some situations which are not covered by [9].

The first step is to give a topological construction of the decomposition diagram (§2). Given a space X , define V_X to be the category whose objects consist of pairs (V, f) , where V is an elementary abelian p -group and f is a homotopy class of maps from BV to X . A map $(V, f) \rightarrow (W, g)$ in V_X is an abelian group map $h: V \rightarrow W$ such that $f = g \circ \bar{h}$, where $\bar{h}: BV \rightarrow BW$ is the homotopy class of maps induced by h . The category A_X is defined to be a subcategory of V_X . An object (V, f) of V_X is in A_X if $V \neq \{0\}$ and the cohomology map f^* makes H^V into a finitely generated module over H^*X ; a map h as above is in A_X if $h: V \rightarrow W$ is a monomorphism of abelian groups.

Let $\text{Map}(BV, X)$ denote the space of maps $BV \rightarrow X$. There is a functor $\hat{\alpha}_X$ from A_X^{op} to the category of spaces which assigns to (V, f) the path component $\text{Map}(BV, X)_f$ of $\text{Map}(BV, X)$ determined by f . (To construct $\hat{\alpha}_X$ it is necessary to settle on some strictly functorial way of constructing BV from V —see, for instance, [14, 23.2, Ch. 3]). Evaluating at the basepoint of BV gives a map $\hat{\alpha}_X(V, f) \rightarrow X$; these maps combine to give a natural map $\text{holim} \hat{\alpha}_X \rightarrow X$. Define $\alpha_X: A_X \rightarrow \mathcal{X}$ to be the functor determined by the formula $\alpha_X(V, f) = H^*(\hat{\alpha}_X(V, f))$.

Let R denote H^*X . Taking mod p cohomology gives a functor $H_X: A_X \rightarrow A_R$, and it is clear from the universal property of $T(V, -)_f$ (cf. 3.1, [10]) that there is a natural transformation $\Gamma_X: \alpha_R \circ H_X \rightarrow \alpha_X$. The following proposition is proved by combining the spectral sequence of [2, XII, 5.8] with a naturality argument.

PROPOSITION 6.1. *Let X be a space and let $R = H^*X$. Suppose that*

- (1) $H_X: A_X \rightarrow A_R$ is an equivalence of categories,
- (2) $\Gamma_X: \alpha_R \circ H_X \rightarrow \alpha_X$ is a natural equivalence, and
- (3) the natural map $R \rightarrow \varprojlim \alpha_R$ is an isomorphism and the groups $\varprojlim^i \alpha_R$ vanish for $i > 0$.

Then the natural map $\text{holim} \hat{\alpha}_X \rightarrow X$ induces an isomorphism on mod p cohomology.

Lannes has provided some mild conditions under which assumptions (1) and (2) of Proposition 6.1 are satisfied. Say that an object R of \mathcal{X} is of *finite type* if each R^k is finite dimensional as an F_p vector space.

PROPOSITION 6.2 [10]. *If X is a p -complete [2] space such that H^*X is of finite type, then H_X is an equivalence of categories.*

PROPOSITION 6.3 [10]. *Let X be a p -complete space such that $R = H^*X$ is of finite type. Assume that for each object $f: V \rightsquigarrow R$ of V_R the graded algebra $T(V, R)_f$ is of finite type and is trivial in dimension 1. Then Γ_X is a natural equivalence.*

Remark. It is also true that assumptions (1) and (2) of 6.1 are satisfied if G is a compact Lie group and $X = BG$ (see 2.2, [11], [6]).

Now let τ be a torus, W a finite group of \mathcal{X} -automorphisms of $H^*B\tau$, and R the ring of invariants $H^*(B\tau)^W$. Suppose that V is an elementary abelian p -group. By the exactness property of the functor $T(V, -)$ [10] there is an isomorphism $T(V, R) \cong T(V, H^*B\tau)^W$, and by [11] the algebra $T(V, H^*B\tau)$ is isomorphic to a direct product of copies of $H^*B\tau$, one for each homomorphism $V \rightarrow \tau$. It follows that for any $f: V \rightarrow R$ the algebra $T(V, R)_f$ is of finite type and is concentrated in even dimensions. In light of 6.1–6.3 and 1.5, this proves the following theorem.

THEOREM 6.4. *Let X be a p -complete space with the property that H^*X is isomorphic to the ring of invariants $H^*(B\tau)^W$ for some torus τ and some finite group W of \mathcal{X} -automorphisms of $H^*B\tau$ (see for instance [3]). Then the natural map $\text{holim} \hat{x}_X \rightarrow X$ induces an isomorphism on mod p cohomology.*

Remark. Sometimes conditions (1) and (2) of 6.1 are satisfied but the map $\text{holim} \hat{x}_X \rightarrow X$ does not induce an isomorphism on mod p cohomology. By more or less explicit calculation this happens if p is 2 and X is the one-point union $B\mathbb{Z}/2 \vee B\mathbb{Z}/2$ — in this case $\text{holim} \hat{x}_X$ is the disjoint union $B\mathbb{Z}/2 \sqcup B\mathbb{Z}/2$. It is not yet clear to us whether or not there is a reasonable characterization of spaces X for which the conclusion of 6.1 holds.

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